CH 1 : ROTATIONAL DYNAMICS

Q. 1 : Define following terms : 1) Linear displacement 2) Linear velocity 3) Linear acceleration 4) Angular displacement 5) angular velocity & 6) angular acceleration.

Ans : 1) Linear displacement : It is the distance travelled by a body in given time.

It is a vector quantity & is denoted by $\vec{s}$.

S.I. Unit => metre (m) Dimensions =>$[L^1M^0T^0]$

2) Linear velocity : It is defined as the distance travelled by a body per unit time in a given direction.

It is a vector quantity. $\vec{v} = \frac{\overrightarrow{ds}}{dt}$

S. I. Unit => m/s Dimensions =>$[L^1M^0T^{-1}]$

3) Linear acceleration : It is defined as rate of change of linear velocity.

It is a vector quantity. $\vec{a} = \frac{\overrightarrow{dv}}{dt}$

S. I. Unit => m/s^2 Dimensions =>$[L^1M^0T^{-2}]$

4) Angular displacement: It is defined as the angle described by radius vector in a given time at the center of circle.

It is a vector quantity. $\theta = \frac{Arc\ length}{radius}$

S. I. Unit => radian (rad), It has No Dimensions.

5) Angular velocity : It is defined as the rate of change of angular displacement.

It is a vector quantity. $\vec{\omega} = \frac{\overrightarrow{d\theta}}{dt}$

S.I. unit => rad/s, Dimensions => $[L^0M^0T^{-1}]$

6) Angular Acceleration : It is defined as the rate of change of angular velocity.

It is a vector quantity. $\vec{\alpha} = \frac{\overrightarrow{d\omega}}{dt}$

S.I. Unit => rad/s^2 , Dimension => $[L^0M^0T^{-2}]$

Q. 2 : What are the characteristics of circular motion?

Ans : 1) It is an accelerated motion – Here Direction of velocity changes at every instant.

2) It is a periodic motion - Here particle repeats the same path in the motion.

Q. 3 : Show that linear speed of particle performing circular motion is the product of radius of circular and angular speed of particle.

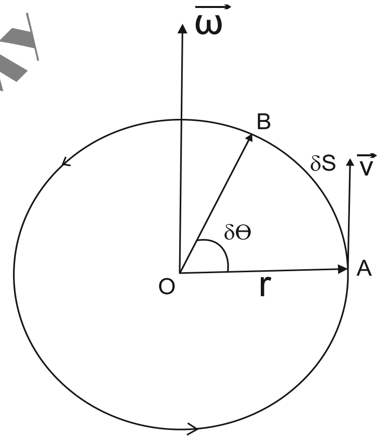
<u>OR</u>

Prove the relation, $\vec{v} = \vec{\omega} \times \vec{r}$ where symbols have their usual meanings.

<u>OR</u>

Find the relation between linear velocity & angular velocity. [2M – Mar'12]

Ans : Consider a particle performing circular motion in a circular path of radius 'r' in anticlockwise sense with centre O. Suppose the particle moves from point A to point B in short time δt. The distance travelled is δs & angular displacement is $\delta\theta$. Let, $\vec{r}$ = position vector. $\vec{\omega}$ = angular velocity & $\vec{v}$ = Linear velocity.



Linear displacement in vector from is given by -

$$\overrightarrow{\delta s} = \overrightarrow{\delta\theta} \times \vec{r}$$

Dividing both sides by δt & Taking limit as $\delta t \to 0$

$$\lim_{\delta t \to 0} \frac{\overrightarrow{\delta s}}{\delta t} = \lim_{\delta t \to 0} \frac{\overrightarrow{\delta\theta}}{\delta t} \times \vec{r}$$

$$\therefore \frac{\overrightarrow{ds}}{dt} = \frac{\overrightarrow{d\theta}}{dt} \times \vec{r}$$

$$\therefore \boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

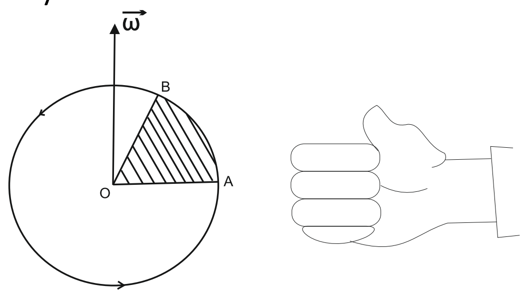
$$\left(\because \frac{\overrightarrow{ds}}{dt} = \vec{v} \ \ \& \ \ \frac{\overrightarrow{d\theta}}{dt} = \vec{\omega}\right)$$

In magnitude, the relation is –

$$\boxed{v = r\,\omega}$$

Q. 4 : State right hand thumb rule which gives direction of angular velocity.

Ans : If the curled fingers of the right hand gives the direction of motion of particle, then outstretched thumb gives direction of angular velocity.



Q. 5 : Define uniform circular motion with example.

Ans : It is defined as the motion of particle along circumference of circle with constant speed.

eg. 1) Motion of earth around the sun
2) Motion of moon around the earth.

***Q*.6 : Define period & frequency of particle performing uniform circular motion. Obtain expression for it. State their S.I. Units.**

Ans : Period : The time taken by a particle performing uniform circular motion (U.C.M) to complete one revolution is called as periodic time or period (T).

$$\text{Period} = \frac{circumference\ of\ circle}{Linear\ velocity}$$

$$\therefore \quad T = \frac{2\pi r}{v}$$

$$\therefore \quad \boxed{T = \frac{2\pi}{\omega}} \qquad (\because v = r\,\omega)$$

S. I. unit => second
Dimension => $[L^0M^0T^1]$

Frequency : It is defined as the number of revolution performed by particle performing uniform circular motion in unit time. It is reciprocal of period of revolution.

$$\therefore \ n = \frac{1}{T}$$

$$\therefore \quad \boxed{n = \frac{\omega}{2\pi} = \frac{V}{2\pi r}}$$

S. I. unit => hertz (Hz)
Dimension => $[L^0M^0T^{-1}]$

Q. 7 : Obtain the relation between magnitude of linear acceleration & angular acceleration in circular motion. [2M – Mar'15]

Ans : Consider a particle performing uniform circular motion (U.C.M) with constant angular velocity 'ω' with path radius 'r'.

Magnitude of linear acceleration is given by

$$\therefore \ a = \frac{dv}{dt}$$

$$= \frac{d}{dt}(r\,\omega) \qquad [\because v = r\,\omega]$$

$$= r\frac{d\omega}{dt} + \omega\left(\frac{dr}{dt}\right)$$

$\because$ r is constant, $\frac{dr}{dt} = 0$

$$\therefore \ a = r\left(\frac{d\omega}{dt}\right)$$

$$\therefore \quad \boxed{a = r\propto} \qquad \left[\because \frac{d\omega}{dt} = \propto\right]$$

In vector from,

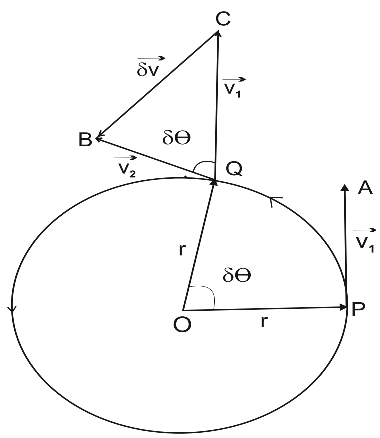
$$\vec{a} = \vec{\propto} \times \vec{r}$$

Q. 8 : Obtain an expression for linear acceleration of a particle performing uniform circular motion.

OR

Find the expression for linear acceleration or radial acceleration or centripetal acceleration in uniform circular motion. [3M – Mar'12]

Ans :



Suppose a particle is performing U.C.M. of radius 'r'. In time δt it moves from P to Q.

Let $\overrightarrow{PA} = \overrightarrow{v_1}$ & $\overrightarrow{QB} = \overrightarrow{v_2}$.

In $\triangle$QCB,

$$\overrightarrow{v_1} + \overrightarrow{\delta v} = \overrightarrow{v_2}$$

$$\therefore \overrightarrow{\delta v} = \overrightarrow{v_2} - \overrightarrow{v_1}$$

From the figure,

$$\angle BQC = \delta\theta$$

If $\delta\theta$ is very small, CB is considered as an arc of circle of radius $v_1 = v_2 = v$.

$$\delta\theta = \frac{\delta V}{v}$$

$$\therefore \delta v = v.\, \delta\theta \qquad \text{--------(i)}$$

Now, $$a = \lim_{\delta t \to 0} \frac{\delta V}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{v\, \delta\theta}{\delta t} \qquad \text{.... From (i)}$$

$$= v \frac{d\theta}{dt}$$

$$\therefore \boxed{a = v\,\omega} \quad \text{......(1)} \qquad [\because \frac{d\theta}{dt} = \omega]$$

$$= (r\,\omega)\,\omega \qquad [\because v = r\,\omega]$$

$$\therefore \boxed{a = r\,\omega^2} \quad \text{.....(2)}$$

$$= r\left(\frac{v}{r}\right)^2 \qquad [\because \omega = \frac{V}{r}]$$

$$\therefore \boxed{a = \frac{v^2}{r}} \quad \text{......(3)}$$

Equations (1) , (2) & (3) are expressions for linear acceleration.

Q. 9 : Define centripetal force. Give its example. [1M – Mar'24]

Ans : It is the force acting on particle performing circular motion, which is along radius of circle & directed towards the centre of circle.

$$\therefore F_{cp} = m\,v\,\omega = m\,r\,\omega^2 = \frac{mv^2}{r}$$

S.I. unit => newton (N)

Dimension => $[L^1M^1T^{-2}]$

e.g. Centripetal force for circular motion of an artificial satellite is provided by gravitational force between earth & satellite.

Q. 10 : Define centrifugal force. Give its example.

Ans : It is a pseudo force in U.C.M, which acts along radius & directed away from the centre of circle.

It has same magnitude as that of centripetal force but direction is opposite.

$$F_{CF} = \frac{mv^2}{r} = mr\omega^2 = m\,v\,\omega$$

e.g. When a moving car along a horizontal curved road takes a turn, passengers in the car experience a force in outward direction. This force is centrifugal.

Q. 11 : Distinguish between centripetal & centrifugal force. [2M – Mar'18]

Ans :

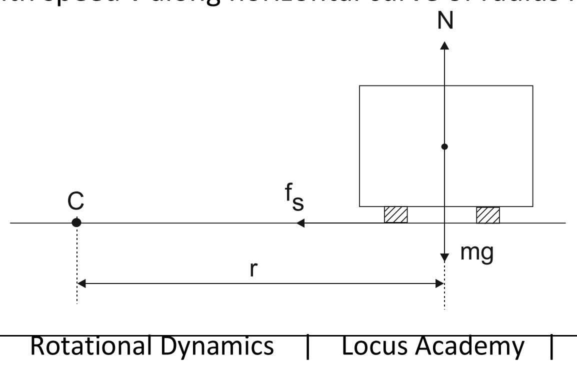
Centripetal force	Centrifugal force
1) It can be defined as the force that causes a body to move in a circular motion.	1) It always tries to push a body away from the center or circular motion.
2) It is real force.	2) It pseudo force.
3) It is directed towards the centre of circular path.	3) It is directed away from the centre of circular path.
4) It is considered in inertial frame of reference.	3) It is considered in non-inertial frame of reference.

Q. 12 : Derive an expression for maximum safety speed with which vehicle should move along a curved horizontal road.

<u>OR</u>

Derive an expression for maximum speed of a vehicle moving along a horizontal circular track. **[2M – Mar'24]**

Ans : Consider a vehicle of mass 'm' moving with speed v along horizontal curve of radius r.



Forces acting on the vehicle are –
1) Weight (mg) - vertically downward
2) Normal (N) - Vertically upwards
3) Force of static friction (f_s) between road and tyres.

Normal balances weight of vehicle & f_s provides necessary centripetal force.

$$N = mg \quad ....(1)$$

$$f_s = \frac{mv^2}{r} \quad .....(2)$$

Equation (2) ÷ Equation (1) =>

$$\therefore \frac{fs}{N} = \frac{\frac{mv^2}{r}}{mg} = \frac{v^2}{rg}$$

Maximum value of $f_s = \mu_s N$
where, μ_s = > Coeff. of static faction

$$\frac{\mu_s N}{N} = \frac{v_{max}^2}{rg}$$

$$\mu_s = \frac{v_{max}^2}{rg}$$

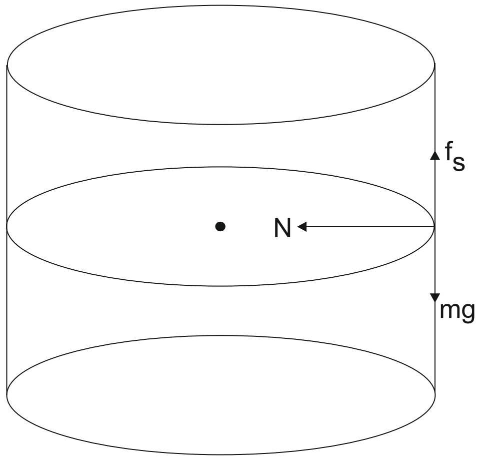
$$\therefore v_{max}^2 = \mu_s rg$$

$$\therefore \boxed{v_{max} = \sqrt{\mu_s rg}}$$

Q. 13 : Find the expression for minimum speed with which vehicle should move in a well of death (wall of death).

Ans : Consider a vertical cylindrical wall of radius r inside which a vehicle is driven in horizontal circles.

Forces acting on the vehicle are –
1) Normal reaction (N) - horizontally towards centre
2) Weight (mg) - Vertically downwards
3) Force of static friction (fs) - Vertically upwards.



Force of friction balances weight of vehicle.
Normal provides necessary centripetal force.

$$N = \frac{mv^2}{r} \quad \& \quad mg = f_s$$

f_s is always less that or equal to μ_s N

$$\therefore f_s <= \mu_s N$$

$$\therefore mg <= \mu_s \left(\frac{mv^2}{r}\right)$$

$$\therefore g <= \frac{\mu_s v^2}{r} \Rightarrow \frac{rg}{\mu_s} <= v^2$$

$$\therefore \boxed{v_{min} = \sqrt{\frac{rg}{\mu_s}}}$$

Q. 14 : What is banking of roads? Why is it necessary?

Ans : The process of raising outer edge of road over its inner edge through certain angle is known is banking of road.

A vehicle moving along a curved road performs circular motion. Therefore it must be provided with centripetal force. Centripetal force is provided by force of friction between surface of road & tyres. Sometimes centripetal force provided by force of friction may not be sufficient. Therefore, there is danger of skidding of vehicle. To avoid this danger, banking of road is done.

Q. 15 : Obtain an expression for maximum safety speed with which vehicle can be safely driven along curved banked road. On what factors Angle of banking depends.
[3M – Mar'12]

OR

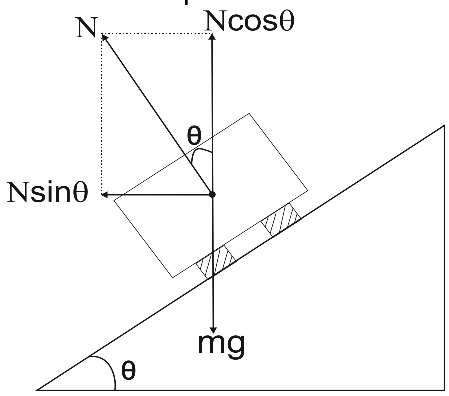
Draw a diagram showing all components of forces acting on a vehicle moving on a curved banked road. Write the necessary equation for maximum safety speed & state the significance of each term involved in it. [3M – Oct'12]

Ans : Suppose a vehicle of mass 'm' moving with speed 'v' on a banked road of radius r banked at angle θ as shown.

Forces on vehicle are -
1) Weight (mg) - vertically downward

2) Normal reaction (N) - Perpendicular to Surface of road.

Normal reaction N can be resolved as -
1) Vertical component = N. cosθ
2) Horizontal component = N. sinθ



N.cos θ balances weight (mg) & N. sin θ provides the necessary centripetal force to vehicle.

$\therefore$ N cos θ = mg ....(1)

$\therefore$ N sin θ = $\frac{mv^2}{r}$ ....(2)

Equation (2)÷ Equation (1) =>

$$\therefore \frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

$$\therefore \boxed{\tan\theta = \frac{v^2}{rg}} \quad ---- (3)$$

1) Most safe speed (Optimum speed) : For a particular road r & θ are fixed. Thus v_s gives most safe speed on this road.

$$\therefore \boxed{v_s = \sqrt{rg\tan\theta}}$$

2) Angle of Banking : From equation (3) =>

$$\therefore \boxed{\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)}$$

Thus angle of banking depends on –
1) Speed of the vehicle
2) Radius of curvature of the road
3) Acceleration due to gravity

Q. 16 : Show that the angle of banking is independent of mass of vehicle.

Ans : Write complete Q. no. (14) upto equation (5).

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As there is no term of mass of vehicle (m) in above equation then angle of banking is independent of mass of (or type of) vehicle.

Q. 17 : Show that the maximum speed or optimum speed of a vehicle along a curved road is independent of mass of vehicle.

Ans : Write complete Q. no. (13).

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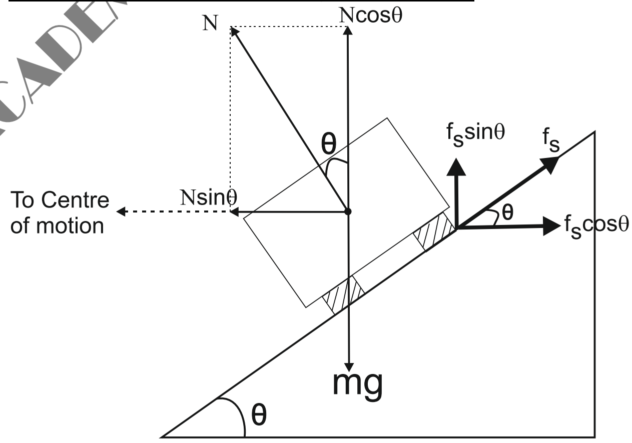
As there is no term of mass of vehicle (m) in above equation, the maximum speed of vehicle is independent of mass of (or type of) vehicle.

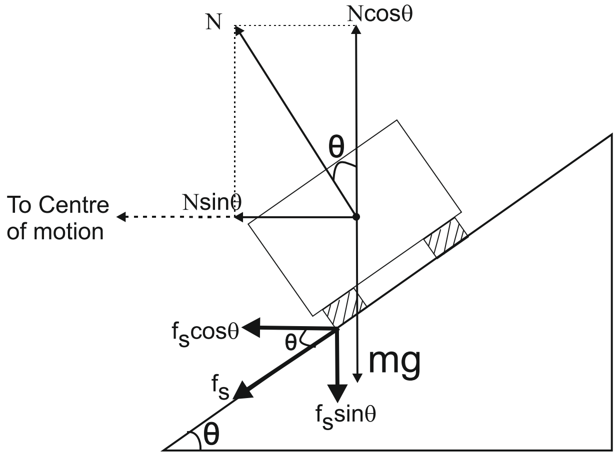
Q. 18 : Find the expression for speed Limits for a vehicle on a banked road. <u>OR</u>
Find the expression for maximum & minimum speed of a vehicle on a banked road.

Ans : Suppose a vehicle of mass 'm' moving with speed 'v' on a banked roads of radius r banked at an angle θ as shown.

<u>1) Banked road (lower speed limit) :</u>



<u>2) Banked road (Upper speed limit) :</u>



If vehicle moves with most safe speed ie. $v_s = \sqrt{rg\tan\theta}$, the forces acting on the vehicle are –

1) Weight (mg) - Vertically downwards
2) Normal reaction (N) - Perpendicular to surface of road.

$N\sin\theta$ component provides necessary centripetal force. But in practice, vehicles never travel exactly with this speed. For other speeds component of force of static friction (f_s) should be considered.

1) Minimum speed :

$$V_1 < \sqrt{rg\tan\theta}$$

$$\frac{mv_1^2}{r} < N\sin\theta$$

Here, direction of f_s between road & tyres is directed along the inclination of the rood, upwards. It's horizontal component $f_s\cos\theta$ is parallel & opposite to $N\sin\theta$. These 2 forces take care of necessary centripetal force.

$$\therefore mg = f_s \sin\theta + N\cos\theta \quad \text{.............(1)}$$

$$\frac{mv_1^2}{r} = N\sin\theta - f_s\cos\theta \quad \text{.............(2)}$$

Equation (2)÷ Equation (1) =>

$$\therefore \frac{\frac{mv_1^2}{r}}{mg} = \frac{N\sin\theta - f_s\cos\theta}{f_s\sin\theta + N\cos\theta}$$

The frictional force $f_s = \mu_s N$

$$\therefore \frac{v_1^2}{rg} = \frac{N\sin\theta - \mu_s N\cos\theta}{\mu_s N\sin\theta + N\cos\theta}$$

Dividing Numerator & Denominator by $N.\cos\theta$

$$\therefore \frac{v_1^2}{rg} = \frac{\tan\theta + \mu_s}{\mu_s\tan\theta + 1}$$

$$\therefore \boxed{v_1 = v_{min} = \sqrt{rg\left[\frac{\tan\theta - \mu_s +}{1+\mu_s\tan\theta}\right]}}$$

2) Maximum speed :

$$V_2 > \sqrt{rg\tan\theta}$$

$$\frac{mv_2^2}{r} > N\sin\theta$$

Here direction of f_s is directed along inclination of the road, downwards. It's horizontal component $f_s\cos\theta$ is parallel to $N\sin\theta$. These 2 forces take care of necessary centripetal force.

$$\therefore mg = N\cos\theta - f_s\sin\theta \quad \text{.............(3)}$$

$$\frac{mv_2^2}{r} = N\sin\theta + f_s\cos\theta \quad \text{.............(4)}$$

On solving these equations –

$$\therefore \boxed{v_2 = v_{max} = \sqrt{rg\left[\frac{\tan\theta + \mu_s}{1-\mu_s\tan\theta}\right]}}$$

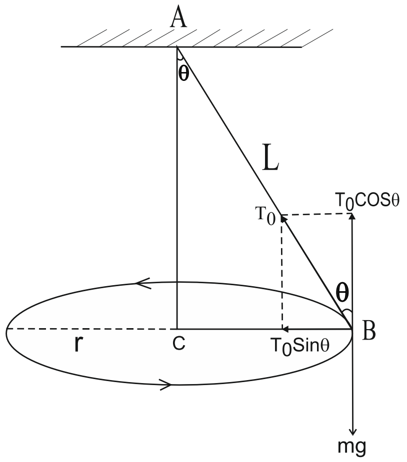
Q. 19 : What is conical pendulum? Define period of conical pendulum and obtain expression for its period.

Ans : Conical pendulum : It is a simple pendulum, which is given such a motion that bob describes a horizontal circle & the string describes a cone.

Period : Time taken by the bob of a conical pendulum to complete on horizontal circle is called time period of conical pendulum.

Q. 20 : Obtain expression for period of conical pendulum.

Ans : Suppose the bob of mass 'm' of conical pendulum with string of length L. Bob performs circular motion in a horizontal circular path of radius r & Centre C. At B, string makes angle θ with vertical.



The forces acting on bob are -

1) Weight (mg) - vertically downward
2) Force T_0 due to Tension- upward along string

The force T_0 can be resolved as -

i) Vertical component $= T_0\cos\theta$

ii) Horizontal component = $T_0 \sin\theta$

$T_0 \cos\theta$ is balanced by weight & $T_0 \sin\theta$ provides necessary centripetal force.

$\therefore \quad T_0 \sin\theta = mr\omega^2$ ....(1)

& $\quad T_0 \cos\theta = mg$ .... (2)

Dividing equation (1) by equation (2) =>

$$\frac{T_0 \sin\theta}{T_0 \cos\theta} = \frac{mr\omega^2}{mg}$$

$$\therefore \ \omega^2 = \frac{g \sin\theta}{r \cos\theta}$$

From figure, $\sin\theta = \frac{r}{L}$ => $r = L \sin\theta$

$$\therefore \ \omega^2 = \frac{g \sin\theta}{L \sin\theta \cos\theta}$$

$$\therefore \ \boxed{\omega = \sqrt{\frac{g}{L \cos\theta}}} \quad \text{------(*)}$$

Period, $T = \frac{2\pi}{\omega}$

$$\therefore \ \boxed{T = 2\pi \sqrt{\frac{L \cos\theta}{g}}}$$

Q. 21 : Derive an expression for angular speed of the bob of conical pendulum.

Ans. : Write upto (*) of Q. No. (20)

Q. 22 : Find the expression for frequency of revolution of a conical pendulum.

Ans : Write complete Q. No. (20)

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Frequency of revolution

$$n = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g}{L \cos\theta}}$$

Q. 23 : What is vertical circular motion? Show that the motion of an object revolving in vertical circle is non-uniform circular motion.

Ans : A body revolving in a vertical circle such that its motion at different points is different then the motion of the body is said to be vertical circular motion.

Consider an object of tied at one end of an inextensible string & whirled in a vertical circle. Due to influence of earth's gravitational field, velocity & tension of the body vary in magnitude from maximum at bottom (lowest) point to minimum at the top (highest) point. Hence motion of an object revolving in vertical circle is non-uniform circular motion.

eg. In a roller coaster, car slows down & speeds up as it moves around vertical loop.

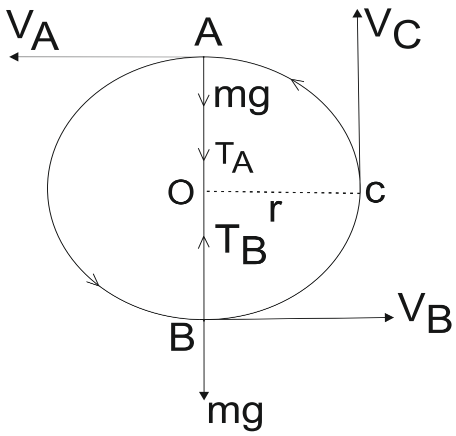
Q. 24 : What is the difference between uniform circular motion & non uniform circular motion (or V.C.M.) ? give examples.

Ans :

U. C. M	Non U.C.M. (V.C.M.)
1) Circular motion with constant angular speed is known as uniform circular motion	1) Circular motion with variable angular speed is called as non- uniform circular motion.
2) For U.C.M , $\alpha = 0$	2) For non-UCM, $\alpha \neq 0$.
3) eg. Motion of the earth around the sun.	3) eg. Motion of a body in vertical circle (Roller coaster).

Q. 25 : Derive expressions for linear velocity at lowest point, midway & top position for a particle revolving in a vertical circle & it has to just complete circular motion without string slackening at top. [4M – Mar'23]

Ans :



Consider an object of mass 'm' tied at the end of string & moves in vertical circle of radius 'r'.

Let v_A = velocity at highest point A

v_B = velocity at lowest point B

& v_C = velocity at midway point C

Let T_A & T_B are Tensions at the highest & lowest point respectively.

1) Velocity at highest point (A) : At the highest point A, the tension in the string together with weight provides necessary centripetal force.

$\therefore T_A + mg = \frac{m{v_A}^2}{r}$

For velocity v_1 at A, string becomes slack ie. Tension vanishes ($T_1 = 0$).

$\therefore 0 + mg = \frac{m{v_A}^2}{r}$

$\therefore {v_A}^2 = rg$

$\therefore \boxed{v_A = \sqrt{rg}}$

2) Velocity at lowest point (B) : The energy possessed by a body at point A is totally K.E.

Using law of conservation of energy -

[K.E. at B] + [P.E. at B] = [K.E. at A] + [P.E. at A]

$\therefore \frac{1}{2} m{v_B}^2 + 0 = \frac{1}{2} m{v_A}^2 + mg(2r)$

{ Total height = 2r & PE =mgh }

Multiplying by $\frac{2}{m}$

$\therefore {v_B}^2 = {v_A}^2 + 4\,rg$

$= rg + 4\,rg$ $\quad (\because v_A = \sqrt{rg})$

$\therefore {v_B}^2 = 5\,rg$

$\therefore \boxed{v_B = \sqrt{5\,rg}}$

3) Velocity at midway point (C) :

Using law of conservation of energy -

[K.E. at B] + [P.E. at B] = [K.E. at C] +[P.E. at C]

$\therefore \frac{1}{2} m{v_B}^2 + 0 = \frac{1}{2} m{v_C}^2 + mgr$

Multiplying by $\frac{2}{m}$

$\therefore {v_B}^2 = {v_C}^2 + 2rg$

$\therefore {v_C}^2 = {v_B}^2 - 2rg$

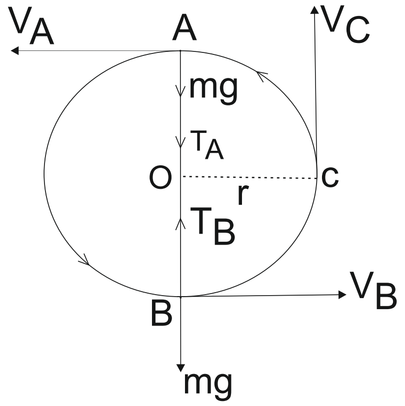
$\therefore {v_C}^2 = 5rg - 2rg$ $\quad (\because v_B = \sqrt{5rg})$

$= 3rg$

$\therefore \boxed{v_C = \sqrt{3rg}}$

Q. 26 : Derive an expression for difference in tension at highest & lowest point for a particle performing vertical circular motion.

Ans : Consider an object of mass 'm' performing vertical circular motion in a vertical circle of radius 'r'.



Let v_A, v_B & v_C be velocities at points A, B & C resp. Let T_A & T_B be Tensions at A & B resp.

At highest point A,

$T_A + mg = \frac{mv_A^2}{r}$

$\therefore T_A = \frac{mv_A^2}{r} - mg$ .... (1)

At lowest point B,

$T_B - mg = \frac{mv_B^2}{r}$

$\therefore T_B = \frac{mv_B^2}{r} + mg$ .... (2)

Equation (2) –Equation (1) =>

$T_B - T_A = \frac{mv_B^2}{r} + mg - [\frac{mv_A^2}{r} - mg]$

$= \frac{mv_B^2}{r} - \frac{mv_A^2}{r} + 2\,mg$

$\therefore T_B - T_A = \frac{m}{r}(v_B^2 - v_A^2) + 2mg$ ....(3)

By law of conservation of energy,

[K.E. at B] + [P.E. at B] = [K.E. at A] + [P.E. at A]

$\therefore \frac{1}{2} m{v_B}^2 + 0 = \frac{1}{2} m{v_A}^2 + mg(2r)$

Multiplying by $\frac{2}{m}$

$\therefore v_B^2 = v_A^2 + 4rg$

$\therefore v_B^2 - v_A^2 = 4rg$

put this value in equation (3) =>

$\therefore T_B - T_A = \frac{m}{r}(4rg) + 2mg$

$= 4mg + 2mg$

$\therefore \boxed{\mathbf{T_B - T_A = 6\ mg}}$

Q. 27 : Obtain expressions for the tensions at highest position, midway position & bottom position for an object revolving in vertical circle.

Ans : Write upto equation (2) from Q. No. 25.

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At midway point C,

$$T_C = \frac{mv_C^2}{r} \quad \text{..............} \quad (3)$$

For the limiting case, when the body is just able to reach to the top of the path & complete the circular motion.

$V_A = \sqrt{rg}$, $v_B = \sqrt{5\,rg}$ & $v_C = \sqrt{3rg}$

Put v_A in equation (1) =>

$$\therefore T_A = \frac{m\,(rg)}{r} - mg$$

$$\therefore \boxed{T_A = 0} \text{ ----------(4) (Highest point)}$$

Put v_B in equation (2) =>

$$\therefore T_B = \frac{m\,(5rg)}{r} + mg$$

$$= 5mg + mg$$

$$\therefore \boxed{T_B = 6mg} \text{ ----------(5) (Lowest point)}$$

Put v_C in equation (3) =>

$$\therefore T_C = \frac{m\,(3rg)}{r}$$

$$\therefore \boxed{T_C = 3mg} \text{ --------(6) (Midway point)}$$

Equations (4), (5) & (6) are the tensions at Highest Point, Lowest & Midway point resp.

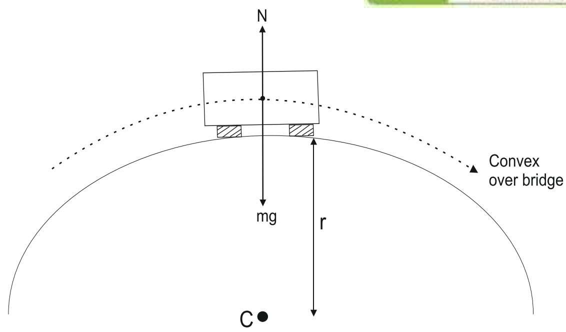
Q. 28 : Derive an expression for maximum velocity of a vehicle at the top of a convex over bridge.

Ans : Consider a vehicle of mass 'm' moving at the top of a convex over bridge during it's motion. Forces acting on vehicle are –

1) Weight (mg) - vertically downward
2) Normal reaction (N) – Vertically upwards

The resultant of these two must provide necessary centripetal force. If v is speed at this point then –

$$mg - N = \frac{mv^2}{r}$$



As speed goes on increasing, N goes on decreasing. Normal reaction is indication of contact. Thus, for just maintaining contact, N=0.

∴ Maximum speed can be found as -

$$mg - 0 = \frac{mv^2}{r}$$

$$\therefore v_{max}^2 = rg$$

$$\therefore \boxed{V_{max} = \sqrt{rg}}$$

Q. 29 : Define moment of inertia. Explain the physical significance of it. [2M-Mar'18]

Ans : Moment of Inertia : M.I. of a rigid body about an axis of rotation is defined as the sum of product of the mass of each particle & the square of its perpendicular distance from the axis of rotation.

$$\therefore I = \sum_{i=0}^{n} m_i r_i^2$$

Physical significance of M.I. :

To produce linear motion in a body, the unbalanced force is applied to overcome its inertia. Here inertia of body is called the mass. The force is -

$$F = m\,a \quad \text{..................} \ (1)$$

To produce rotational motion, torque is applied to overcome its inertia. Here, inertia of body is called moment of inertia (I). The torque is -

$$\tau = I \propto \quad \text{...................} \ (2)$$

From equations (1) & (2) it is clear that M.I. plays same role in rotational motion as the mass of the body does in linear motion.

Q. Define moment of inertia of a rotating rigid body. State its SI unit and dimensions. [2M-Mar'22]

Ans : Definition – Q. No. 29

S.I. unit = kgm^2 , Dimensions = $[L^2 M^1 T^0]$
It is a scalar quantity.

Q. 30 : What do you mean by radius of gyration of a body? Write it's S.I. unit & dimensions.

Ans : The radius of gyration of a body about an axis of rotation is the distance of axis of rotation from a point where whole mass of body is supposed to be concentrated so as to possess same M.I. as that of body.

Radius of gyration of a body depends on distribution of mass of body about axis of rotation & independent of mass of body. It is denoted by K.

S.I. unit = m

Dimensions = $[M^0 L^1 T^0]$

Q. 31 : State & explain physical significance of radius of gyration. OR
Why is it useful to define radius of gyration.

Ans : 1) If the particles of the body are distributed close to the axis of rotation, the radius of gyration is less.
2) If the particles are distributed away from the axis of rotation, the radius of gyration is more.

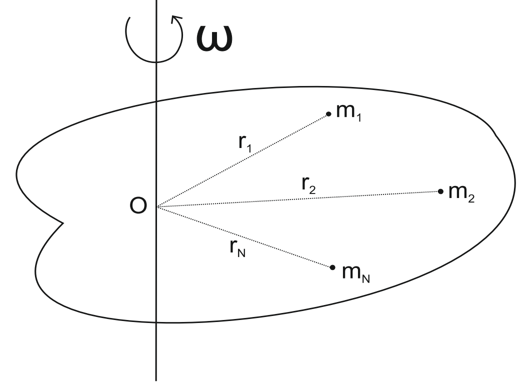
The knowledge of mass and radius of gyration of the body about a given axis of rotation gives the value of its moment of inertia about the same axis, even if we do not know the actual shape of the body.

$$I = \sum_{i=1}^{n} m_i r_i^2 = M K^2$$

Q. 32 : Derive an expression for kinetic energy of a rotating body (Rotational K.E.) moving with uniform angular velocity. OR
Derive an expression for kinetic energy of a rotating body. [3M – July'22, 3M – Mar'22]

Ans : Suppose a rigid body is rotating with constant angular velocity 'ω' about an axis passing through point 'O'. The axis of rotation is perpendicular to plane of paper.

Suppose the body is made of n particles of masses m_1 , m_2 , m_3 , …………….. m_N situated at distance r_1 , r_2 , r_3 ……….. r_N resp. from the axis of rotation.



All the particles perform circular motion with same angular velocity 'ω'. The linear velocities of particles are different. Suppose v_1 , v_2 , v_3 , ………… V_N are their linear velocities.

Translational K.E. of first particle is -

$$K.E._1 = \frac{1}{2} m_1 v_1^2$$
$$= \frac{1}{2} m_1 r_1^2 \omega^2 \qquad (v = r\omega)$$

Similarly K.E. of remaining particles are -

$$K.E._2 = \frac{1}{2} m_2 r_2^2 \omega^2$$
$$K.E._3 = \frac{1}{2} m3\, r_3^2 \omega^2$$
.
.
.
$$K.E._n = \frac{1}{2} m_n r_n^2 \omega^2$$

The Rotational K.E. of body is -

$$KE_{rotational} = K.E._1 + K.E._2 + K.E._3 + \ldots\ldots\ldots + K.E._n$$
$$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots\ldots + \frac{1}{2} m_n r_n^2 \omega^2$$

$$= \frac{1}{2} [m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots\ldots\ldots\ldots + m_n r_n^2]\, \omega^2$$
$$= \frac{1}{2} [\sum_{i=1}^{n} m_i r_i^2]\, \omega^2$$

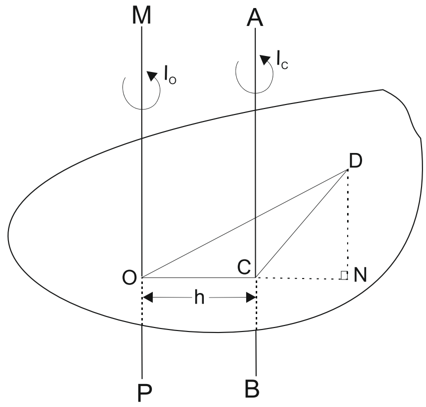
$$\therefore \boxed{KE_{rotation} = \frac{1}{2} I \omega^2} \quad \ldots [\because \sum_{i=1}^{n} m_i r_i^2 = I]$$

Q. 33: State & prove principle of parallel axes. [4M – Mar'14, 4M – July'23]

Ans : The M.I. of object about any axis is the sum of its M.I. (I_C) about a parallel axis passing thro' its centre of mass & the product of its mass & square of the perpendicular distance between the two parallel axes (Mh^2).

$$I_O = I_c + Mh^2$$

Proof :



Consider an object of mass ‘M’. Axis MOP is any axis thro’ O. Axis ACB is passing through the centre of mass C of the object, parallel to axis MOP & at distance h from it.

Consider a mass element dm located at point D. Perpendicular on OC produced from point D is DN.

M.I. of object about the axis ACB is –

$\therefore I_c = \int (DC)^2.dm$

M.I. of object about the axis MOP is –

$\therefore I_0 = \int (DO)^2.dm$

$= \int [(DN)^2+(NO)^2].dm$

$= \int [(DN)^2+(NC+CO)^2].dm$

$= \int [(DN)^2+(NC)^2+2(NC)(CO)+(CO)^2].dm$

$= \int [(DC)^2 + 2(NC)(h) + h^2].dm$

-----{ $(DN)^2+(NC)^2 = (DC)^2$

$=\int (DC)^2dm + 2h\int NC.dm + h^2\int dm$

Now, $\int (DC)^2dm = Ic$ & $\int dm = M$

NC is the distance of a point from the centre of mass. Any mass distribution is symmetric about the centre of mass. Thus, from definition of centre of mass -

$\int NC.dm = 0$

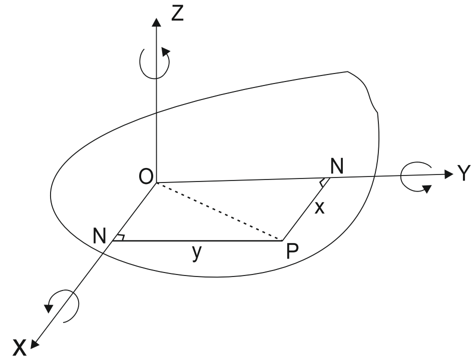
$\therefore$ $\boxed{I_O = I_c + Mh^2}$

Q. 34 : State & prove principle of perpendicular axes.

Ans : The M.I. (I_z) of a laminar object about an axis perpendicular to it's plane is the sum of it's moment of inertias about two mutually perpendicular axes (x & y) in it's plane, all the three axes being concurrent.

$I_Z = I_x + I_y$

Proof :



Consider a rigid laminar object able to rotate about 3 mutually perpendicular axes x, y & z. Axes x & y are in plane of the object while z-axis is perpendicular to it & all are concurrent at O.

Consider a mass element dm located at any point P. PM = y & PN = x are perpendicular drawn from P resp. on x & y axes.

If I_x, I_y & I_Z are respective Moment of Inertias about x, y & z axes resp.

$I_x = \int PM^2.dm = \int y^2.dm$

$I_y = \int PN^2.dm = \int x^2.dm$

$I_Z = \int OP^2.dm$

$= \int (y^2 + x^2).dm$

$= \int y^2.dm + \int x^2.dm$

$\therefore$ $\boxed{I_Z = I_x + I_y}$

Q. 35 : State theorem of parallel axes & theorem of perpendicular axes about moment of Inertia. [2M – Mar’15]

Ans: Write only statements of two theorems.

Q. 36 : Define angular momentum of a body. Obtain an expression for angular momentum of a rigid body rotating with uniform angular velocity. State it’s SI Unit & dimensions.

<u>OR</u>

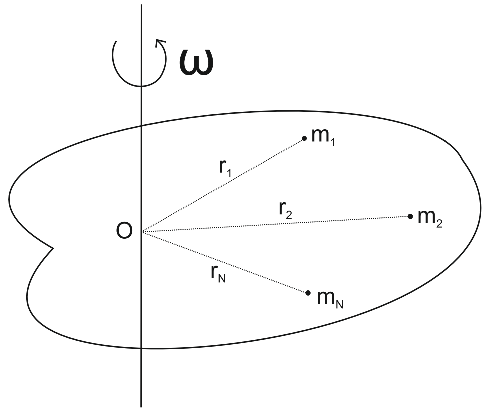
Derive an expression that relates angular momentum with the angular velocity of a rigid body.

Ans : Angular momentum of a body is product of M.I. of body about the axis & it's angular velocity.

$$L = I\omega$$

S.I. unit = kg.m^2/s , Dimensions = $[L^2 M^1 T^{-1}]$

Expression for angular momentum :



Suppose a rigid object is rotating with constant angular speed ω about an axis perpendicular to the plane of paper.

Suppose the object is made of a n particles of masses m_1, m_2, m_3 .... m_N situated at distances r_1 , r_2 , r_3 .... r_N resp. from the axis of rotation.

As the object rotates, all these particles perform UCM with same angular speed ω but with different linear speeds v_1, v_2, v_3 .... V_N.

The linear momentum of first particle -

$$p_1 = m_1 v_1$$

$\therefore \quad p_1 = m_1 r_1 \omega$ ......($v = r\omega$)

The angular momentum -

$\therefore L_1 = p_1 \times r_1$

$= m_1 r_1 \omega \times r_1$

$\therefore L_1 = m_1 r_1^2 \omega$

Similarly, angular momentum of remaining particles are -

$L_2 = m_2 r_2^2 \omega$

$L_3 = m_3 r_3^2 \omega$

.

.

.

$L_n = m_n r_n^2 \omega$

Angular momentum of body is sum of angular momentum of all particles.

$\therefore$ $L = L_1 + L_2 + L_3$ .... + L_n

$= m_1 r_1^2\omega + m_2 r_2^2\omega + ... + m_n r_n^2\omega$

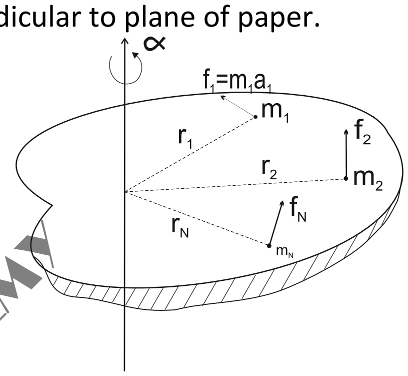
$= (m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2)\,\omega$

$= [\sum_{i=1}^{n} m_i r_i^2]\ \omega$

$\therefore$ $\boxed{L = I\omega}$ ...... $[\because \sum_{i=1}^{n} m_i r_i^2 = I]$

Q. 37 : Obtain an expression relating the torque with angular acceleration for a rigid body.

Ans : Suppose a rigid object is rotating with constant angular acceleration α about an axis perpendicular to plane of paper.



Suppose the object is made of a n particles of masses m_1, m_2, m_3 .... m_N situated at distances r_1 , r_2 , r_3 .... r_N resp. from the axis of rotation.

As the object rotates, all these particles perform circular motion with same angular acceleration α but with different linear accelerations a_1, a_2, a_3 .... a_N.

The force experienced by first particle is –

$f_1 = m_1 a_1$

$= m_1 r_1 \alpha$ ---- ($a = r\alpha$)

Torque experienced by it is -

$\tau_1 = f_1 \times r_1$

$\therefore\ \tau_1 = m_1 r_1^2 \propto$

Similarly, the torque on remaining particles are -

$\tau_2 = m_2 r_2^2 \propto$

$\tau_3 = m_3 r_3^2 \propto$

.

.

.

$\tau_N = m_N r_N^2 \propto$

The resultant torque on body is

$\tau = \tau_1 + \tau_2 + \tau_3 + ... + \tau_N$

$= m_1 r_1^2 \propto + m_2 r_2^2 \propto + \ldots\ldots\ldots\ldots + m_N r_N^2 \propto$

$= [m_1 r_1^2 + m_1 r_2^2 + \ldots\ldots\ldots\ldots + m_N r_N^2]\propto$

$= [\sum_{i=1}^{n} m_i r_i^2] \propto$

$\therefore \boxed{\tau = I \propto}$ ...... $[\because \sum_{i=1}^{n} m_i r_i^2 = I]$

Q. 38 : State & prove principle of conservation of angular momentum. Explain it with examples. <u>OR</u>
State & prove the theorem of conservation of angular momentum. [3M-Mar'18, 2M–Oct'14]

Ans : The angular momentum of a rotating body is <u>conserved or remains constant</u>, if the resultant external torque acting on the body is <u>zero</u>.

i.e. If $\vec{\tau} = 0$ then $\frac{\overrightarrow{dL}}{dt} = 0$ <u>or</u> $\vec{L}$ = constant.

Proof : Consider a particle rotating about an axis with torque τ. Let $\vec{p}$ be the linear momentum of the particle of mass 'm' & $\vec{r}$ be its position vector.

Angular momentum of body is -

$\therefore \vec{L} = \vec{r} \times \vec{p}$

Diff. w. r. t. time

$\therefore \frac{\overrightarrow{dL}}{dt} = \frac{\vec{d}}{dt}(\vec{r} \times \vec{p})$

$= \vec{r} \times \frac{\overrightarrow{dP}}{dt} + \vec{p} \times \frac{\overrightarrow{dr}}{dt}$ ....(1)

$= \vec{r} \times \vec{F} + m\vec{v} \times \vec{v}$

$(\because \frac{\overrightarrow{dP}}{dt} = \vec{F},\ \vec{p} = m\vec{v}\ \&\ \frac{\overrightarrow{dr}}{dt} = \vec{v}$

$= \vec{r} \times \vec{F} + m(\vec{v} \times \vec{v})$

$= \vec{r} \times \vec{F} + 0$ $(\because \vec{v} \times \vec{v} = 0)$

$= \vec{\tau}$ $(\because \vec{r} \times \vec{F} = \vec{\tau})$

$\therefore \vec{\tau} = \frac{\overrightarrow{dL}}{dt}$

Thus if $\vec{\tau} = 0$ then $\frac{\overrightarrow{dL}}{dt} = 0$ <u>or</u> $\vec{L}$ = constant.

Thus if no unbalanced external torque acts on a body then angular momentum of body remains constant.

e.g. A spinning (gyrating) skater, acrobat etc. increases the speed of rotation by folding his hands & legs close to his body.

Q. 39 : Deduce an expression for kinetic energy when a body is rolling on a plane surface without slipping.

Ans : The rolling object possesses two types of kinetic energies, rotational and translational. Sum of these two is Total K.E.

Consider an object of Moment of Inertia I, rolling uniformly.

Let v = Linear speed of centre of mass
R = Radius of body
ω = Angular velocity = $\frac{V}{R}$
M = Mass of the body
K = Radius of gyration of body = MK^2

Total rolling K.E.=Translational K.E. +Rotational K.E.

$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$E = \frac{1}{2}MV^2 + \frac{1}{2}(MK^2)\frac{V^2}{R^2}$

$E = \frac{1}{2}MV^2 + \frac{1}{2}\frac{M K^2 V^2}{R^2}$

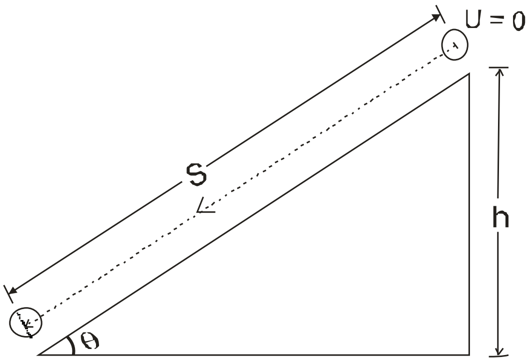
$\therefore \boxed{E = \frac{1}{2}Mv^2 (1 + \frac{K^2}{R^2})}$

Q. 40 : A rigid object is rolling down an inclined plane. Derive expressions for the acceleration along the track and the speed after falling through a certain vertical distance.

Ans : Consider a rigid object of mass 'M' & radius 'R' rolling down an inclined plane of inclination θ.

As the object starts rolling down, it's gravitational P.E. is converted into K.E. of rolling. Starting from rest, let v be the speed of centre of mass as the object comes down through a vertical distance h.

From law of conservation of energy,
$\{\text{K. E. initial}\} + \{\text{P. E. initial}\} =$
$\{\text{K. E. final}\} + \{\text{P. E. final}\}$



$\therefore 0 + Mgh = \left[\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2\right] + 0$

$= \frac{1}{2}Mv^2 + \frac{1}{2}MK^2.\left(\frac{v}{R}\right)^2$

$(\because I = MK^2 \ \& \ \omega = \frac{v}{R})$

$\therefore Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{MK^2v^2}{R^2}$

$\therefore Mgh = \frac{1}{2}Mv^2 \left[1 + \frac{K^2}{R^2}\right]$

$\therefore v^2 = \frac{2gh}{1+K^2/R^2}$

$\therefore \boxed{v = \sqrt{\frac{2gh}{1+K^2/R^2}}}$

From figure, $\sin\theta = \frac{h}{s}$ $\therefore s = \frac{h}{\sin\theta}$

Now, $v^2 = u^2 + 2as$

$2as = v^2 - u^2$

$2a\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{1+K^2/R^2} - 0$

$\therefore \boxed{a = \frac{g\sin\theta}{1+K^2/R^2}}$

FORMULAE

1) Motion :

	Displacement	Velocity	Acceleration
Linear	s	$v = \frac{ds}{dt}$	$a = \frac{dv}{dt}$
Angular	θ	$\omega = \frac{d\theta}{dt}$	$\alpha = \frac{d\omega}{dt}$
Relation	$S = r\theta$	$v = r\omega$	$a = r\alpha$

2) Acceleration :

Tangential	Centripetal (Radial)	Total
$a_T = r\alpha$ (In UCM, $a_T = 0$)	$a_r = \frac{v^2}{r}$	$a = \sqrt{a_T^2 + a_r^2}$

3) Period : $T = \frac{2\pi}{\omega} = \frac{1}{n}$

4) Centripetal force : $F_{cp} = \frac{mv^2}{r} = mr\omega^2$

5) Speed :

Horizontal curved road	Well of Death	Convex Bridge	Banked Road	Banked road to avoid slipping [Max]
$v = \sqrt{\mu rg}$	$v = \sqrt{\frac{rg}{\mu}}$	$v = \sqrt{\mu rg}$	$v = \sqrt{rg\tan\theta}$	$v = \sqrt{Rg\left[\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}\right]}$

6) M.I. :

Basic	Rod	Ring (Cylinder)	Disc (Flywheel)	Sphere
$I = \sum m_i r_i^2 = MK^2$	$I = \frac{ML^2}{12}$	$I = MR^2$	$I = \frac{MR^2}{2}$	$I = \frac{2MR^2}{5}$

7) Conical Pendulum : $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

8) V.C.M. :

Point	Velocity	Tension	Energy
Highest	$v = \sqrt{rg}$	$T = \frac{mv^2}{r} + mg\cos\theta$ (θ => Zero at lowest)	$E = \frac{2mgr}{5}$
Midway	$v = \sqrt{3rg}$		
Lowest	$v = \sqrt{5rg}$		

9) K.E. :

Rotation	Translation	Rolling
$KE_{rot} = \frac{1}{2} I\omega^2$	$KE_{tran} = \frac{1}{2} Mv^2$	$KE_{rolling} = \frac{1}{2}[Mv^2 + I\omega^2] = \frac{1}{2} Mv^2\left[1 + \frac{K^2}{R^2}\right]$

10) M.I. about axes :

Parallel	Perpendicular
$I_o = I_c + Mh^2$	$I_z = I_x + I_y$

11) **Torque :** $\tau = I\alpha$

12) **Angular Momentum :** $L = I\omega$

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Q.1 : A motorcyclist performs stunt along the cylindrical wall of a 'Well of Death' of inner radius 4 m. Coefficient of static friction between the tyres and the wall is 0.4. Calculate the maximum period of revolution. [Use $g = 10\ m/s^2$] [2M – July'23]

Q.2 : Calculate the moment of inertia of a uniform disc of mass 10 kg and radius 60 cm about an axis perpendicular to its length and passing through its centre. [2M – Mar'22]

Q. 3 : A motor cyclist (to be treated as point mass) is to undertake horizontal circles inside the cylindrical wall of well of inner radius 4m. The co-efficient of static friction between tyres and wall is 0.2. Calculate the minimum speed and period necessary to perform this stunt. [2M – Mar'21]

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